/ informatik & security

# Ackermann function

## Theoretical Computer Science

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# 1 Ackermann function

In 1926 David Hilbert (German mathematician, 1826-1943) conjectured that every computable function is primitive-recursive. A few weeks later in the same year Wilhelm Ackermann (German mathematician, 1896-1962) constructed a computable function that is not primitive-recursive and published it in 1928. Afterwards, Ackermann himself and also some other mathematicians modified this function, so that there are now many functions, all called "Ackermann function", but all showing the same behavior.

The function shown here is one of these functions, so it is an Ackermann function:

## 1.1 Definition

The function is a binary function, so it has two input parameters, $a$ and $b$, both of them are positive integers, and also the values of this function are positive integers so we can write it as

$$A(a,b): \quad \mathbb{N}^2 \longrightarrow \mathbb{N}, \;\; a,b > 0$$

We can draw the values in a grid, similar to the grid of a spreadsheet program:

| b / a | 1 | 2 | 3 | 4 | 5 | 6 | ⋯ |
|---|---|---|---|---|---|---|---|
| 1 | A(1,1) | A(1,2) | A(1,3) | A(1,4) | A(1,5) | A(1,6) | ⋯ |
| 2 | A(2,1) | A(2,2) | A(2,3) | A(2,4) | A(2,5) | A(2,6) | ⋯ |
| 3 | A(3,1) | A(3,2) | A(3,3) | A(3,4) | A(3,5) | A(3,6) | ⋯ |
| 4 | A(4,1) | A(4,2) | A(4,3) | A(4,4) | A(4,5) | A(4,6) | ⋯ |
| 5 | A(5,1) | A(5,2) | A(5,3) | A(5,4) | A(5,5) | A(5,6) | ⋯ |
| 6 | A(6,1) | A(6,2) | A(6,3) | A(6,4) | A(6,5) | A(6,6) | ⋯ |
| 7 | A(7,1) | A(7,2) | A(7,3) | A(7,4) | A(7,5) | A(7,6) | ⋯ |
| 8 | A(8,1) | A(8,2) | A(8,3) | A(8,4) | A(8,5) | A(8,6) | ⋯ |
| 9 | A(9,1) | A(9,2) | A(9,3) | A(9,4) | A(9,5) | A(9,6) | ⋯ |
| 10 | A(10,1) | A(10,2) | A(10,3) | A(10,4) | A(10,5) | A(10,6) | ⋯ |
| 11 | A(11,1) | A(11,2) | A(11,3) | A(11,4) | A(11,5) | A(11,6) | ⋯ |
| 12 | A(12,1) | A(12,2) | A(12,3) | A(12,4) | A(12,5) | A(12,6) | ⋯ |
| 13 | A(13,1) | A(13,2) | A(13,3) | A(13,4) | A(13,5) | A(13,6) | ⋯ |
| 14 | A(14,1) | A(14,2) | A(14,3) | A(14,4) | A(14,5) | A(14,6) | ⋯ |
| 15 | A(15,1) | A(15,2) | A(15,3) | A(15,4) | A(15,5) | A(15,6) | ⋯ |
| 16 | A(16,1) | A(16,2) | A(16,3) | A(16,4) | A(16,5) | A(16,6) | ⋯ |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋱ |

Note, that this grid has a left border and a top border, but no right or bottom border, because it is infinitely large in these two directions.

Depending on how many and to which borders a cell is adjacent, we can define 4 different types of cells:

| a \ b | 1 | 2 | 3 | 4 | 5 | 6 | ... |
|---|---|---|---|---|---|---|---|
| 1 | A(1,1) | A(1,2) | A(1,3) | A(1,4) | A(1,5) | A(1,6) | ... |
| 2 | A(2,1) | A(2,2) | A(2,3) | A(2,4) | A(2,5) | A(2,6) | ... |
| 3 | A(3,1) | A(3,2) | A(3,3) | A(3,4) | A(3,5) | A(3,6) | ... |
| 4 | A(4,1) | A(4,2) | A(4,3) | A(4,4) | A(4,5) | A(4,6) | ... |
| 5 | A(5,1) | A(5,2) | A(5,3) | A(5,4) | A(5,5) | A(5,6) | ... |
| 6 | A(6,1) | A(6,2) | A(6,3) | A(6,4) | A(6,5) | A(6,6) | ... |
| 7 | A(7,1) | A(7,2) | A(7,3) | A(7,4) | A(7,5) | A(7,6) | ... |
| 8 | A(8,1) | A(8,2) | A(8,3) | A(8,4) | A(8,5) | A(8,6) | ... |
| 9 | A(9,1) | A(9,2) | A(9,3) | A(9,4) | A(9,5) | A(9,6) | ... |
| 10 | A(10,1) | A(10,2) | A(10,3) | A(10,4) | A(10,5) | A(10,6) | ... |
| 11 | A(11,1) | A(11,2) | A(11,3) | A(11,4) | A(11,5) | A(11,6) | ... |
| 12 | A(12,1) | A(12,2) | A(12,3) | A(12,4) | A(12,5) | A(12,6) | ... |
| 13 | A(13,1) | A(13,2) | A(13,3) | A(13,4) | A(13,5) | A(13,6) | ... |
| 14 | A(14,1) | A(14,2) | A(14,3) | A(14,4) | A(14,5) | A(14,6) | ... |
| 15 | A(15,1) | A(15,2) | A(15,3) | A(15,4) | A(15,5) | A(15,6) | ... |
| 16 | A(16,1) | A(16,2) | A(16,3) | A(16,4) | A(16,5) | A(16,6) | ... |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋱ |

### 1.1.1 4 types of cells

Now, let's define something that gives the cells a concrete value:

- The red cell (there is only one, it is in the top left corner) gets a constant value:

$$A(1,1) = 2$$

| a \ b | 1 | 2 | 3 | 4 | 5 | 6 | ... |
|---|---|---|---|---|---|---|---|
| 1 | 2 | A(1,2) | A(1,3) | A(1,4) | A(1,5) | A(1,6) | ... |
| 2 | A(2,1) | A(2,2) | A(2,3) | A(2,4) | A(2,5) | A(2,6) | ... |
| 3 | A(3,1) | A(3,2) | A(3,3) | A(3,4) | A(3,5) | A(3,6) | ... |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋱ |

- The green cells (those at the top border) get their values from their left neighbors:

$$\text{for all } b > 1\text{: } A(1, b) \;=\; A(1, b-1)$$

| a \ b | 1 | 2 | 3 | 4 | 5 | 6 | … |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 2 | 2 | 2 | 2 | 2 | … |
| 2 | A(2,1) | A(2,2) | A(2,3) | A(2,4) | A(2,5) | A(2,6) | … |
| 3 | A(3,1) | A(3,2) | A(3,3) | A(3,4) | A(3,5) | A(3,6) | … |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋱ |

- The yellow cells (those at the left border) are the only cells for which we have to perform a calculation, but this calculation is not very hard: Take the value from the upper neighbor and add the value of the red cell (which is 2):

$$\text{for all } a > 1\text{: } A(a, 1) \;=\; A(a-1, 1) + A(1, 1)$$

| a \ b | 1 | 2 | 3 | 4 | 5 | 6 | … |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 2 | 2 | 2 | 2 | 2 | … |
| 2 | 4 | A(2,2) | A(2,3) | A(2,4) | A(2,5) | A(2,6) | … |
| 3 | 6 | A(3,2) | A(3,3) | A(3,4) | A(3,5) | A(3,6) | … |
| 4 | 8 | A(4,2) | A(4,3) | A(4,4) | A(4,5) | A(4,6) | … |
| 5 | 10 | A(5,2) | A(5,3) | A(5,4) | A(5,5) | A(5,6) | … |
| 6 | 12 | A(6,2) | A(6,3) | A(6,4) | A(6,5) | A(6,6) | … |
| 7 | 14 | A(7,2) | A(7,3) | A(7,4) | A(7,5) | A(7,6) | … |
| 8 | 16 | A(8,2) | A(8,3) | A(8,4) | A(8,5) | A(8,6) | … |
| 9 | 18 | A(9,2) | A(9,3) | A(9,4) | A(9,5) | A(9,6) | … |
| 10 | 20 | A(10,2) | A(10,3) | A(10,4) | A(10,5) | A(10,6) | … |
| 11 | 22 | A(11,2) | A(11,3) | A(11,4) | A(11,5) | A(11,6) | … |
| 12 | 24 | A(12,2) | A(12,3) | A(12,4) | A(12,5) | A(12,6) | … |
| 13 | 26 | A(13,2) | A(13,3) | A(13,4) | A(13,5) | A(13,6) | … |
| 14 | 28 | A(14,2) | A(14,3) | A(14,4) | A(14,5) | A(14,6) | … |
| 15 | 30 | A(15,2) | A(15,3) | A(15,4) | A(15,5) | A(15,6) | … |
| 16 | 32 | A(16,2) | A(16,3) | A(16,4) | A(16,5) | A(16,6) | … |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋱ |

A shorter way to define the values in the yellow cells might have been this:

$$A(a, 1) \;=\; 2a$$

This in fact reproduces all values correctly, but this is not how the function is defined! There is a big difference in the time needed to calculate the value for, let's say $A(28513,1)$. The handy formula with the multiplication needs just $2$ look-ups and $1$ multiplication, but the operation from the actual definition needs $28513$ look-ups $28512$ additions.

- All the magic of the Ackermann function happens in the blue cells:

$$\text{for all } a > 1 \text{and for all } b > 1{:}\ A(a, b) = A(A(a-1, b), b-1)$$

What does this mean?

It means, that the value of $A(a, b)$ is copied from a cell somewhere in the column $b-1$:

$$A(a, b) = A(A(a-1, b), b-1)$$

And the row number of this source is determined by the value of the upper neighbor $A(a-1, b)$.

## 1.2 Values

Let's find some values, step by step:

What is the value for $A(2,2)$? It is the value of $A(A(2-1,2), 2-1) = A(A(1,2), 1)$. So, we need to find the value of $A(1,2)$ and this is $2$. And this is the number of the row. This means:

$$A(2,2) = A(2,1)$$

And there we find the value $4$. So: $A(2,2) = 4$

| a \ b | 1 | 2 | 3 | 4 | 5 | 6 | ⋯ |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 2 | 2 | 2 | 2 | 2 | ⋯ |
| 2 | 4 | 4 | A(2,3) | A(2,4) | A(2,5) | A(2,6) | ⋯ |
| 3 | 6 | A(3,2) | A(3,3) | A(3,4) | A(3,5) | A(3,6) | ⋯ |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋱ |

Let's try $A(2,3)$:

$$A(2,3) = A(A(2-1,3), 3-1)$$

$$A(2,3) = A(A(1,3), 2)$$

$$A(2,3) = A(2,2)$$

$$A(2,3) = 4$$

Let's try $A(2,4)$:

$$A(2,4) = A(A(2-1,4), 4-1)$$

$$A(2,4) = A(A(1,4), 3)$$

$$A(2,4) = A(2,3)$$

$$A(2,4) = 4$$

Do you see the pattern?

| a \ b | 1 | 2 | 3 | 4 | 5 | 6 | … |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 2 | 2 | 2 | 2 | 2 | … |
| 2 | 4 | 4 | 4 | 4 | 4 | 4 | … |
| 3 | 6 | A(3,2) | A(3,3) | A(3,4) | A(3,5) | A(3,6) | … |
| 4 | 8 | A(4,2) | A(4,3) | A(4,4) | A(4,5) | A(4,6) | … |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋱ |

Do you have an idea, which values will be in the other cells?

Let's start with the next row:

$$A(a,b) = A(A(a-1,b),b-1)$$

$$A(3,2) = A(A(3-1,2),2-1)$$

$$A(3,2) = A(A(2,2),1)$$

$$A(3,2) = A(4,1)$$

$$A(3,2) = 8$$

| a \ b | 1 | 2 | 3 | 4 | 5 | 6 | … |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 2 | 2 | 2 | 2 | 2 | … |
| 2 | 4 | 4 | 4 | 4 | 4 | 4 | … |
| 3 | 6 | 8 | A(3,3) | A(3,4) | A(3,5) | A(3,6) | … |
| 4 | 8 | A(4,2) | A(4,3) | A(4,4) | A(4,5) | A(4,6) | … |
| 5 | 10 | A(5,2) | A(5,3) | A(5,4) | A(5,5) | A(5,6) | … |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋱ |

Before we continue this row, let's see what comes below this cell, at $A(4,2)$ and $A(5,2)$:

$$A(a,b) = A(A(a-1,b),b-1)$$

$$A(4,2) = A(A(4-1,2),2-1)$$

$$A(4,2) = A(A(3,2),1)$$

$$A(4,2) = A(8,1)$$

$$A(4,2) = 16$$

$$A(5,2) = A(A(5-1,2),2-1)$$

$$A(5,2) = A(A(4,2),1)$$

$$A(5,2) = A(16,1)$$

$$A(5,2) = 32$$

| a \ b | 1 | 2 | 3 | 4 | 5 | 6 | … |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 2 | 2 | 2 | 2 | 2 | … |
| 2 | 4 | 4 | 4 | 4 | 4 | 4 | … |
| 3 | 6 | 8 | A(3,3) | A(3,4) | A(3,5) | A(3,6) | … |
| 4 | 8 | 16 | A(4,3) | A(4,4) | A(4,5) | A(4,6) | … |
| 5 | 10 | 32 | A(5,3) | A(5,4) | A(5,5) | A(5,6) | … |
| 6 | 12 | A(6,2) | A(6,3) | A(6,4) | A(6,5) | A(6,6) | … |
| 7 | 14 | A(7,2) | A(7,3) | A(7,4) | A(7,5) | A(7,6) | … |
| 8 | 16 | A(8,2) | A(8,3) | A(8,4) | A(8,5) | A(8,6) | … |
| 9 | 18 | A(9,2) | A(9,3) | A(9,4) | A(9,5) | A(9,6) | … |
| 10 | 20 | A(10,2) | A(10,3) | A(10,4) | A(10,5) | A(10,6) | … |
| 11 | 22 | A(11,2) | A(11,3) | A(11,4) | A(11,5) | A(11,6) | … |
| 12 | 24 | A(12,2) | A(12,3) | A(12,4) | A(12,5) | A(12,6) | … |
| 13 | 26 | A(13,2) | A(13,3) | A(13,4) | A(13,5) | A(13,6) | … |
| 14 | 28 | A(14,2) | A(14,3) | A(14,4) | A(14,5) | A(14,6) | … |
| 15 | 30 | A(15,2) | A(15,3) | A(15,4) | A(15,5) | A(15,6) | … |
| 16 | 32 | A(16,2) | A(16,3) | A(16,4) | A(16,5) | A(16,6) | … |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋱ |

Obviously, the values of the cells in column 2 double from row to row. So,

$$\text{for all } a > 1: A(a,2) = 2 \cdot A(a-1,2)$$

Or, in an even simpler formula:

$$A(a,2) = 2^a$$

But again: This is just a handy (but still correct) formula to quickly calculate the values. This is not how these values are really defined.

Now we have this image of the Ackermann function:

| a \ b | 1 | 2 | 3 | 4 | 5 | 6 | … |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 2 | 2 | 2 | 2 | 2 | … |
| 2 | 4 | 4 | 4 | 4 | 4 | 4 | … |
| 3 | 6 | 8 | A(3,3) | A(3,4) | A(3,5) | A(3,6) | … |
| 4 | 8 | 16 | A(4,3) | A(4,4) | A(4,5) | A(4,6) | … |
| 5 | 10 | 32 | A(5,3) | A(5,4) | A(5,5) | A(5,6) | … |
| 6 | 12 | 64 | A(6,3) | A(6,4) | A(6,5) | A(6,6) | … |
| 7 | 14 | 128 | A(7,3) | A(7,4) | A(7,5) | A(7,6) | … |
| 8 | 16 | 256 | A(8,3) | A(8,4) | A(8,5) | A(8,6) | … |
| 9 | 18 | 512 | A(9,3) | A(9,4) | A(9,5) | A(9,6) | … |
| 10 | 20 | 1024 | A(10,3) | A(10,4) | A(10,5) | A(10,6) | … |
| 11 | 22 | 2048 | A(11,3) | A(11,4) | A(11,5) | A(11,6) | … |
| 12 | 24 | 4096 | A(12,3) | A(12,4) | A(12,5) | A(12,6) | … |
| 13 | 26 | 8192 | A(13,3) | A(13,4) | A(13,5) | A(13,6) | … |
| 14 | 28 | 16384 | A(14,3) | A(14,4) | A(14,5) | A(14,6) | … |
| 15 | 30 | 32768 | A(15,3) | A(15,4) | A(15,5) | A(15,6) | … |
| 16 | 32 | 65536 | A(16,3) | A(16,4) | A(16,5) | A(16,6) | … |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋱ |

Let's do the next cell, $A(3,3)$:

$$A(a,b) = A(A(a-1,b),b-1)$$
$$A(3,3) = A(A(3-1,3),3-1)$$
$$A(3,3) = A(A(2,3),2)$$
$$A(3,3) = A(4,2)$$
$$A(3,3) = 16$$

And the cell below, $A(4,3)$:

$$A(a,b) = A(A(a-1,b),b-1)$$
$$A(4,3) = A(A(4-1,3),3-1)$$
$$A(4,3) = A(A(3,3),2)$$
$$A(4,3) = A(16,2)$$
$$A(4,3) = 65536$$

One more, $A(5,3)$:

$$A(a,b) = A(A(a-1,b),b-1)$$
$$A(5,3) = A(A(5-1,3),3-1)$$
$$A(5,3) = A(A(4,3),2)$$
$$A(5,3) = A(65536,2)$$

At this point we have to select the value of a cell in row 65536. This is where the exponentiation formula, we found before, comes handy:

$$A(a,2) = 2^a$$

$$A(5,3) = A(65536,2) = 2^{65536}$$

A(5,3) =
2003529930406846464979072351560255750447825475569751419265016973710894059556311453089506130880933348101038234342907263181822949382118812668869506364761547029165041871916351587966347219442930927982084309104855990570159318959639524863372367203002916969592156108764948889254090805911457037675208500206671563702366126359747144807111774815880914135742720967190151836282560618091458852699826141425030123391108273603843767876449043205960379124490905707560314035076162562476031863793126484703743782954975613770981604614413308692118102485959152380195331030292162800160568670105651646750568038741529463842244845292537361442533614373729088303794601274724958414864915930647252015155693922628180691650796381064132275307267143998158508811292628901134237782705567421080070065283963322155077831214288551675554073345107213112427399562982719769150054883905223804357045848197956393157853510018992000024141963706813559840464039472194016069517690156119726982337890017641517190051133466306898140219383481435426387306539552969691388024158161859561100640362119796101859534802787167200122604642492385111393400464

51623867567078745259464670903886547743483217897012764455529409092021
95958575162297333357615955239488529757995402847194352991354376370598
69289137571537400019863943324648900525431066296691652434191746913896
32476560289415199775477703138064781342309596190960654591300890188887
58808473362595606544488850144733570605881709016210849971452956834406
19796905654698136311620535793697914032363284962330464210661362002201
75787851857409162050489711781820400187282939943446186224328009837323
76493181478984811945271300744022076568091037620399920349202390662626
44919091679854615157788390603977207592793788522412943010174580868622
63369284725851403039615558564330385450688652213114813638408384778263
79045960718687672850976347127198889068047824323039471865052566097815
07298611414303058169279249714091610594171853522758875044775922183011
58780701975535722241400019548102005661773589781499532325208589753463
54700778669040642901676380816174055040511767009367320280454933902799
24918673065399316407204922384748152806191669009338057321208163507076
34351669869625020969023162859350071874190579161241536897514808261904
84794657173660100589247665544584083833479054414481768425532720731558
63493476051374197795251903650321980201087647383686825310251833775339
08861426184800374008082238104076468878471647552945326947661700424461
06331123802113458869453220011656407632702307429242605158281107038701
83453245676356259514300320374327407808790562836634069650308442258559
67039271869461158513793386475699748568670079823960604393478850861649
26030494506174341236582835214480672667684180708375486221140823657980
29612000274413244384324023312574035450193524287764308802328508558860
89962774458164680857875115807014743763867976955049991643998284357290
41537814343884730348426190338884149403136613985425763557710533558020
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This number has 19729 decimal digits. It's simpler to write it as $2^{65536}$.

| b / a | 1 | 2 | 3 | 4 | 5 | 6 | ⋯ |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 2 | 2 | 2 | 2 | 2 | ⋯ |
| 2 | 4 | 4 | 4 | 4 | 4 | 4 | ⋯ |
| 3 | 6 | 8 | 16 | A(3,4) | A(3,5) | A(3,6) | ⋯ |
| 4 | 8 | 16 | 65536 | A(4,4) | A(4,5) | A(4,6) | ⋯ |
| 5 | 10 | 32 | $2^{65536}$ | A(5,4) | A(5,5) | A(5,6) | ⋯ |
| 6 | 12 | 64 | A(6,3) | A(6,4) | A(6,5) | A(6,6) | ⋯ |
| 7 | 14 | 128 | A(7,3) | A(7,4) | A(7,5) | A(7,6) | ⋯ |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋱ |

So, we have in column 3:

$$A(1,3) = 2$$

$$A(2,3) = 4 = 2^2$$

$$A(3,3) = 16 = 2^4 = 2^{2^2}$$

$$A(4,3) = 65536 = 2^{16} = 2^{2^{2^2}}$$

$$A(5,3) = 2^{65536} = 2^{2^{2^{2^2}}}$$

$$A(a,3) = \underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2^2}}}}}}_{\text{number of 2s} = a}$$

There is no closed formula for $A(a, 3)$. Mathematicians have invented a new notation to write this:

$$A(a, 3) = 2 \uparrow a$$

The order of which the values in column 3 are growing is called "superexponential".

We have seen, that $A(5,3)$ has almost 20000 decimal digits. $A(6,3)$ is already so extremely huge, that only printing the number of digits would cover another 7 pages of this document. If you would print the value of $A(6,3)$ on paper, you would need more matter for producing the ink, than there is matter in the visible universe. The ink immediately would collapse to a black whole by it's own gravitation. I think it is justified to call this number big.

$A(7,3)$ is even bigger.

| a \ b | 1 | 2 | 3 | 4 | 5 | 6 | ⋯ |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 2 | 2 | 2 | 2 | 2 | ⋯ |
| 2 | 4 | 4 | 4 | 4 | 4 | 4 | ⋯ |
| 3 | 6 | 8 | 16 | A(3,4) | A(3,5) | A(3,6) | ⋯ |
| 4 | 8 | 16 | 65536 | A(4,4) | A(4,5) | A(4,6) | ⋯ |
| 5 | 10 | 32 | $2^{65536}$ | A(5,4) | A(5,5) | A(5,6) | ⋯ |
| 6 | 12 | 64 | big | A(6,4) | A(6,5) | A(6,6) | ⋯ |
| 7 | 14 | 128 | bigger | A(7,4) | A(7,5) | A(7,6) | ⋯ |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋱ |

Well, what is the value of $A(3, 4)$?

$$A(a, b) = A(A(a - 1, b), b - 1)$$
$$A(3, 4) = A(A(3 - 1, 4), 4 - 1)$$
$$A(3, 4) = A(A(2, 4), 3)$$
$$A(3, 4) = A(4, 3)$$
$$A(3, 4) = 65536$$

| a \ b | 1 | 2 | 3 | 4 | 5 | 6 | ⋯ |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 2 | 2 | 2 | 2 | 2 | ⋯ |
| 2 | 4 | 4 | 4 | 4 | 4 | 4 | ⋯ |
| 3 | 6 | 8 | 16 | 65536 | A(3,5) | A(3,6) | ⋯ |
| 4 | 8 | 16 | 65536 | A(4,4) | A(4,5) | A(4,6) | ⋯ |
| 5 | 10 | 32 | $2^{65536}$ | A(5,4) | A(5,5) | A(5,6) | ⋯ |
| 6 | 12 | 64 | big | A(6,4) | A(6,5) | A(6,6) | ⋯ |
| 7 | 14 | 128 | bigger | A(7,4) | A(7,5) | A(7,6) | ⋯ |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋱ |

Now, let's think about $A(4,4)$:
The value of $A(4,4)$ is the value of $A(65536,3)$. Remember the 7 pages needed to print $A(6,3)$? And how mind-blowing big $A(7,3)$ was? And now imagine doing the same steps again and again more than 65,000 times. And at the end you have the value of $A(65536,3)$ which is the value of $A(4,4)$. And, if you think sharply, you will see, that this also is the value of $A(3,5)$

| a \ b | 1 | 2 | 3 | 4 | 5 | ⋯ |
|---|---|---|---|---|---|---|
| 1 | 2 | 2 | 2 | 2 | 2 | ⋯ |
| 2 | 4 | 4 | 4 | 4 | 4 | ⋯ |
| 3 | 6 | 8 | 16 | 65536 | Holy crap, is this big! | ⋯ |
| 4 | 8 | 16 | 65536 | Holy crap, is this big! | A(4,5) | ⋯ |
| 5 | 10 | 32 | $2^{65536}$ | A(5,4) | A(5,5) | ⋯ |
| 6 | 12 | 64 | big | A(6,4) | A(6,5) | ⋯ |
| 7 | 14 | 128 | bigger | A(7,4) | A(7,5) | ⋯ |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋱ |

All other numbers are so incredible huge that it no longer makes sense to try to describe them in any way. The human brain is simply not capable of imagining such insanely large numbers.

# 2 A unary Ackermann function

The function described in the previous chapter is a binary function. This means, it has two input values, $a$ and $b$, that define together the value of the function. But for many cases it would be nice to have a unary function that behaves like the function described in chapter 1. But such a function is easy to define. Let's call it $A'$:

$$A'(x) := A(x,x)$$

The new unary Ackermann function contains all values in the diagonal of our previous function:

| b / a | 1 | 2 | 3 | 4 | 5 | 6 | ⋯ |
|---|---|---|---|---|---|---|---|
| 1 | **A′(1)** | A(1,2) | A(1,3) | A(1,4) | A(1,5) | A(1,6) | ⋯ |
| 2 | A(2,1) | **A′(2)** | A(2,3) | A(2,4) | A(2,5) | A(2,6) | ⋯ |
| 3 | A(3,1) | A(3,2) | **A′(3)** | A(3,4) | A(3,5) | A(3,6) | ⋯ |
| 4 | A(4,1) | A(4,2) | A(4,3) | **A′(4)** | A(4,5) | A(4,6) | ⋯ |
| 5 | A(5,1) | A(5,2) | A(5,3) | A(5,4) | **A′(5)** | A(5,6) | ⋯ |
| 6 | A(6,1) | A(6,2) | A(6,3) | A(6,4) | A(6,5) | **A′(6)** | ⋯ |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋱ |

$$A'(1) = 2$$

$$A'(2) = 4$$

$$A'(3) = 16$$

$$A'(4) = \text{Holy crap, is this big!}$$

$$\vdots$$

# 3 Why do we need such functions?

The point is, that both functions shown here (and all other Ackermann functions) have well-defined values for every single argument (or for every single pair of arguments). This means, these functions are computable.

But there is no simple formula that allows you to calculate the values of these function. We could find simple formulas for column 1 and for column 2 of the binary Ackermann function:

$$A(a, 1) \;=\; 2a$$

$$A(a, 2) = 2^a$$

We even could define a new notation for column 3:

$$A(a, 3) = 2 \uparrow a$$

And this arrow-notation also helps us to write short formulas for other columns:

$$A(a, 4) = 2 \uparrow\uparrow a$$

$$A(a, 5) = 2 \uparrow\uparrow\uparrow a$$

etc.

But this is just a notation. It doesn't help us to calculate the values, and this is meant when mathematicians say:

The Ackermann function is not primitive-recursive computable.

## 3.1 primitive-recursive functions

Before we can define primitive recursive functions, we need 3 basic functions:

- constant functions
  They always have the same value, no matter what value their argument(s) has/have.
  Example:
  $f(x) = 4$ for all values of x
- projections
  A projection selects one value out of a tuple
  Example:
  We have this tuple: $t = (4, 1, 17, 5, 29)$ and the projection-function $\pi_i(t)$. Then we have:
  $\pi_1(t) = 4,\ \ \pi_2(t) = 2,\ \ \pi_3(t) = 17,$ etc.
- successor functions
  returns the next natural number
  Example:
  $S(1) = 2,\ S(2) = 3,\ S(85) = 86,$ etc.

These function are primitive recursive by definition, but also any function you can build by composing other primitive recursive functions are again primitive recursive functions.

David Hilbert thought, that anything that is computable can be describes as a primitive recursive function, but Wilhelm Ackermann showed, that this is not true, and the functions named after him are examples for functions that are not primitive-recursive.

## 3.2 $\mu$-recursive functions

The greek letter mu ($\mu$) stands for *μικρότατος* which means "the smallest". These kind of functions are all functions that are computable in an intuitive sense. There is also a strict mathematical definition, but it is out of scope of this course. Important for us is another definition:

Any function that is $\mu$-recursive can be computed by a Turing machine.

The set of $\mu$-recursive functions is equal to the set of Turing-computable functions.

All Ackermann functions are Turing-computable and therefor $\mu$-recursive functions.

But there are other functions that are even crazier than Ackermann functions. The busy-beaver-function is not $\mu$-recursive which also means, it is not Turing-computable. It is impossible to find a Turing machine that can compute the values of the busy beaver function, although all values of this function are natural numbers which are well defined.